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Using Open-Response Fraction Items to Explore the Relationship Between Instructional Modalities and Students' Solution Strategies

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Abstract

The purpose of this study was to explore the relationship between instructional modality used for teaching fractions and third- and fourth-grade students' responses and strategies to open-response fraction items. The participants were 155 third-grade and 200 fourth-grade students from 17 public school classrooms. Students within each class were randomly assigned to two instructional treatment groups: a virtual manipulatives representations (VMR) instruction group and a physical manipulatives and textbook representations (PMTR) instruction group. A conversion mixed methods analysis was used to examine quantitative and qualitative data. The quantitative analysis showed achievement outcomes were the same for both groups. The qualitative analysis revealed shifts in learning that were otherwise hidden with solely quantitative achievement results. Specifically, the results indicated VMR group success in understanding fractions as relationships and PMTR group success in maintaining conceptualization of the whole. Overall, the results of this study corroborate previous research indicating the importance of both types of instructional modalities, showing that virtual manipulatives and physical manipulatives are effective instructional tools with positive effects on student learning. The study expands existing research by offering an opportunity to explore the nuances of students' fractions understanding and provide a window into students' shifts in fraction learning.

Introduction

Elementary teachers use a variety of instructional modalities when teaching children early fraction concepts. Their instruction often includes physical, pictorial, and symbolic representations. Some teachers use virtual manipulatives (Moyer, Bolyard, & Spikell, 2002), which combine representations (e.g., pictorial and symbolic) and representational modalities (e.g., visual and haptic). Studies indicate that using multiple representations and modalities in fraction instruction develops and expands students' understanding of fractions (Behr, Lesh, Post, & Silver, 1983; Moyer-Packenham & Westenskow, 2013; Sowell, 1989).

The purpose of this study was to explore the relationship between instructional modalities used for learning fraction concepts—specifically using virtual manipulatives or physical manipulatives with textbooks—and students' solution strategies on open-response fraction items. We employed a conversion mixed methods approach (Teddlie & Tashakkori, 2006) to analyze open-response items, which we coded and quantitized for quantitative and qualitative analysis. Open-response items provide windows into students' thinking processes and strategies for solving mathematics tasks (Cai, 2000; Cai, Magone, Wang, & Lane, 1996; Lane, 1993). This study complements and extends previous studies by using open-response items to examine these phenomena in depth using qualitative analysis with a large sample of participants ($n = 355$).

The study was framed as a comparison between the learning outcomes of two groups of students using different modalities for learning fraction concepts (i.e., virtual and physical manipulatives). As you will read, our Mann-Whitney *U* analysis corroborated prior research (e.g., Burns & Hamm, 2011; Manches et al., 2010; Melideo & Dodson, 2009; Mendiburo & Hasselbring, 2011; Moyer-Packenham et al., 2013) indicating no numerical achievement differences between the groups. Hence, in the paper, we aimed to explore the more nuanced patterns in students' responses and strategies through a qualitative analysis. We selected specific student work

examples for the Results section to highlight patterns and interesting features of students' responses and strategies on these open-response items.

The examples we selected highlight key themes that emerged in our analyses, namely, shifts in learning from pretest to posttest and small differences between the groups' responses and strategies. Furthermore, through the process of this rigorous qualitative analysis of 355 students' strategies, we developed a classification scheme of the strategies that emerged (see Appendix A), which we anticipate will be helpful to the research community.

Representations, Instructional Modalities, and Fraction Learning

As children develop their understandings of number and quantities from whole numbers to rational numbers, they often struggle with understanding that a fraction represents a relationship. Children have difficulty understanding the meaning of the denominator, keeping track of the whole, and thinking multiplicatively (Behr & Post, 1992; Kamii & Clark, 1995; Smith, 2002). To help children overcome these challenges, representations are often at the heart of teaching and learning the persistently difficult concept of fractions.

Research (e.g., Cramer, Post, & delMas, 2002; Sowell, 1989) and mathematics learning theories (e.g., Bruner, 1966; Cobb, 1995) emphasize the important role played by various conceptual representations in students' learning. Representations include signs, symbols, models, images, or objects that stand for a particular reality (Cai, 2005; Goldin & Shteingold, 2001) and are used to mediate and express learning. Representations can be used as an instructional aid to make sense of mathematics and to externally represent and express students' internal mental models of mathematics. Cai (2005) termed pedagogical representations as those representations used by teachers and students to explain and learn concepts. Solution representations "are the visible records generated by a solver to communicate thinking of the solution processes" (p.137). In the current study, pedagogical representations were considered during fraction instruction and learning, while solution representations were considered in the analyses of students' pictorial and symbolic responses to open-response test items.

Pedagogical Representations

Pedagogical representations are often categorized as physical, pictorial, or symbolic. Various pedagogical representations illuminate different aspects of a fraction concept. Students need a variety of representations to support their understanding of fraction concepts (National Council of Teachers of Mathematics, 2000). Physical representations typically include physical manipulative models that a student can touch, handle, and manipulate to explore a mathematical concept. Fraction circles and fraction bars are two common examples of physical manipulatives. Pictorial representations are non-animated pictures, which provide a visual image that relates to physical examples, such as a region-model drawing that illustrates ¾ of a pie. Examples of symbolic representations include numerals, words, and equations.

When learning fractions, students often struggle with symbolic representations, specifically, understanding symbolic fraction notation and the meaning of the numerator and denominator (Behr & Post, 1992). Bruner (1966) proposed that these pedagogical representations help children make sense of their world when used in stages: first through enactive means (i.e., the manipulation of physical objects) that then connect with iconic (visual images, pictures) and symbolic (words, numbers, symbols) representations. The obvious advantage of physical manipulatives in mathematics learning is the concrete action of physically manipulating objects to learn mathematics concepts. Different from physical manipulatives, virtual manipulatives often provide students opportunities to work physically with iconic representations. Additionally, one of the unique advantages of virtual manipulatives is that they often directly link iconic and symbolic representations.

Using Physical Manipulatives to Learn Fraction Concepts

Sowell's (1989) meta-analysis of 60 studies on the effectiveness of mathematics instruction with physical manipulatives indicated that physical manipulatives were most effective when compared to symbolic-only instruction and when physical manipulatives were used long-term. A recent review of manipulatives by Carbonneau, Marley, and Selig (2012) identified 55 studies that compared physical manipulatives-based instruction to a control condition of abstract mathematics symbols-based instruction and found small to moderate effect sizes in favor of the physical manipulatives-based instruction. Carbonneau et al. (2012)

extended Sowell's (1989) meta-analysis by identifying moderators of physical manipulatives' effectiveness (e.g., an object's perceptual richness, level of guidance during learning, and students' age).

Using Virtual Manipulatives to Learn Fraction Concepts

Clements and McMillian (1996) suggested possibilities for thinking outside of the typically designated categories of physical, pictorial, and symbolic because technology provides new ways of thinking about what is "concrete" or "physical." For example, many virtual manipulatives, defined as "an interactive, Web-based visual representation of a dynamic object" (Moyer, Bolyard, & Spikell, 2002, p. 373), do not fit neatly into distinct categories and often combine representations and modalities. By engaging with virtual manipulatives, students leave the concrete portion of Bruner's concrete-pictorial-abstract model (1966) and go beyond the pictorial phase, because virtual manipulatives provide a dynamic visual or pictorial model (Moyer, Niezgoda, & Stanley, 2005). A recent meta-analysis of 82 effect size scores from 32 studies comparing the effects of using virtual manipulatives on student achievement with other methods of instruction indicates that virtual manipulatives have moderate effects on student achievement during instruction when compared to other types of instruction (Moyer-Packenham & Westenskow, 2013).

Using Physical and Virtual Manipulatives to Learn Fraction Concepts

Several recent studies compare elementary students' achievement when using physical manipulatives versus virtual manipulatives to learn fractions (e.g., Burns & Hamm, 2011; Manches, O'Malley, & Benford, 2010; Melideo & Dodson, 2009; Mendiburo & Hasselbring, 2011). These studies indicate that physical manipulatives and virtual manipulatives are equally effective for fraction instruction. For example, Burns and Hamm (2011) randomly assigned 91 third-grade students to complete a lesson using either physical or virtual manipulatives to learn fraction concepts. Using a pretest-posttest design, Burns and Hamm (2011) found that both types of manipulatives were effective in teaching third-grade students fraction concepts.

Moyer-Packenham and Westenskow's (2013) meta-analysis reported that when virtual and physical manipulatives are combined during instruction and compared with other instructional treatments, there are moderate effects on student achievement. The meta-analysis results indicated that both virtual manipulatives alone and virtual manipulatives combined with physical manipulatives have instructional features that positively impact students' mathematics achievement. The implications of these results were interpreted through the lens of embodied knowledge, which proposes that students' interactions with, and connections among, multiple embodiments of mathematics concepts aids students' learning of abstract concepts (Dienes, 1973; Lakoff & Nunez, 2000). The present study represents an important extension of the existing research. Rather than using only pretest and posttest multiple-choice scores, as many studies have done in the past, the present study looks more closely at the test items that were open-response. By examining and coding 355 third- and fourth-grade students' responses and strategies on open-response fraction items, this study takes an in-depth look at the patterns that emerged from hundreds of students' solutions and strategies when different instructional modalities were used.

Purpose and Research Questions

The purpose of this study was to explore the relationship between instructional modalities and students' solution strategies on open-response fraction items. Participants were enrolled in a larger study aimed at examining differences in achievement and variables that predict performance when manipulatives are used for mathematics instruction. Detailed descriptions about the larger study are discussed in separate publications (see Moyer-Packenham et al., 2013; Moyer-Packenham et al., 2014). In the current study, we explored the following research question: What is the relationship between instructional modality (virtual manipulatives or physical manipulatives with text-based materials) and students' solution strategies on open-response fraction tasks?

Method

Participants and Setting

Participants were 155 third-grade and 200 fourth-grade students from 17 public school classrooms in 7 different elementary schools located in 2 school districts in the western United States. Students were assigned to one of

two treatment groups (virtual manipulatives representations or physical manipulatives and text-based representations) using within-class random assignment. Teachers who taught the Virtual Manipulatives Representations (VMR) groups used primarily virtual manipulatives representations to teach fraction concepts during a 2-3 week unit of instruction. Teachers who taught the Physical Manipulatives and Textbook Representations (PMTR) groups used primarily physical manipulatives and text representations to teach the 2-3 week unit of instruction.

Data Source and Data Collection

The main source of data in this project was the open-response fraction items, which were two of the 19 items that students completed on pretests, posttests, and delayed-posttests in the larger project. The open-response items on these tests asked students to draw a picture and/or write an explanation to justify their solutions. Unlike the larger study, which sought to examine overall achievement and predictive variables, this examination took an in-depth look at students' solutions to open-response items. Researchers used the open-response items to assess students' responses and representations, identify students' errors, and examine students' strategies. Using open-response items to assess students' mathematical reasoning and strategies reveals aspects of students' thinking beyond the correct/incorrect information provided by multiple-choice questions (Cai, 1995; Cai, 2000; Silver, 1992).

Each open-response fraction item selected for this analysis was a matched question (i.e., similar or the same) that appeared on the pretest and posttest. There were two sets of matched questions for third grade and two sets of matched questions for fourth grade. The two third-grade open-response items that appeared on both the pretest and posttest were the Chocolate Bar task and the Candy Cane task. These two tasks required students to draw a picture (a fraction model) to explain their solutions. In fourth grade, two open-response items that appeared on the pretest and posttest were the Comparing Fractions task and the String task. The Comparing Fractions task asked students to draw a picture using the context of a candy bar (region model). The String task asked students to partition and shade the given whole to create equivalent fractions (length model).

Data Analysis

Researchers analyzed the open-response data using a conversion mixed methods approach (Teddlie & Tashakkori, 2006). In this analysis method, researchers first used qualitative coding techniques to analyze students' various types of responses and strategies, then quantified the coded qualitative data by assigning numerical codes for each response-type and strategy-type, and finally summarized the data using quantitative and qualitative methods.

Coding and Major Categories

Researchers used an iterative interpretation process of students' responses, coding, analysis, discussion, and reconsideration to define codes, categories, scores, and descriptions of students' errors and strategies (Cai, Lane, & Jakabcsin, 1996; Cai et al., 1996). To ensure inter-rater reliability, pairs of coders conducted the analyses together for the open-response questions. The first phase of inductive analysis involved establishing major categories based on emergent themes in students' responses (Patton, 1990). Pairs of coders interpreted, analyzed, and coded students' responses for correct solutions, incorrect solutions, and themes that fit between the correct and incorrect categories (e.g., partially correct answers, error patterns, strategies). Two coders evaluated and coded each student's responses together. When coding differed between the two coders, a consensus was reached through discussion and/or a third coder's analysis. Coders assigned numerical codes to students' responses (see Appendix A). Tables and graphs were used to compare frequencies of responses and strategies, visually analyze the responses of each group, and compare the students' errors and strategies between the VMR and PMTR groups of students.

Reconsideration

In the second phase of qualitative analysis, all questions were coded again for either errors or strategies. Recoding occurred because our initial frames were limited and further analysis was warranted (Miles & Huberman, 1994). Specifically, the String task was recoded for *errors* because strategies were a stronger theme

in the initial phase of coding, while the other three items were recoded for *strategies* because errors were a stronger theme in the initial phase of coding.

Data Summary

Finally, researchers compiled and summarized the data, focusing on trends in students' responses and differences in strategies between the two treatment groups. The data were not normally distributed, hence, a non-parametric Mann-Whitney *U* analysis was conducted to test for variance in responses (including students' errors) and strategies between the VMR and PMTR groups. Cross-tabulation frequency tests assisted researchers in locating qualitative differences in the frequencies of solutions and strategies between the VMR and PMTR groups.

Tables and graphs were used as tools to summarize the patterns and trends in the data and to facilitate the process of creative synthesis (Patton, 1990). Creative synthesis entailed "bringing together of the pieces that have emerged into a total experience, showing patterns and relationships" of instructional modalities, students' responses to the tasks, and their strategies for solving the fraction tasks which assisted us in interpreting the results of the study (Patton, 1990, p. 410).

Results

Our research question focused on the relationship between the representational modalities (VMR or PMTR) used to learn fraction concepts and students' responses and strategies to open-response fraction items. A Chi-square test of the distribution of types of responses ($\chi^2 = 134.26$) and types of strategies ($\chi^2 = 189.91$) indicated that neither were normally distributed. Therefore, individual Mann-Whitney *U* tests were used to assess the differences in strategies and responses in the VM and PMT groups.

Results of these tests indicated that the groups were similar in regards to types of responses ($p = .966$) and use of strategies ($p = .413$). The Mann-Whitney *U* corroborated prior research (e.g., Burns & Hamm, 2011; Manches et al., 2010; Melideo & Dodson, 2009; Mendiburo & Hasselbring, 2011; Moyer-Packenham et al., 2013) indicating no numerical achievement differences between the groups, hence researchers in this project aimed to explore the more nuanced patterns in students' responses and strategies.

Researchers selected specific student work examples for the Results section to highlight patterns and interesting features of students' responses and strategies on these open-response items. The examples that were selected highlight key themes that emerged in our analyses, namely, shifts in learning from pretest to posttest and small differences between the VMR and PMTR groups' responses and strategies. This section is organized around each of the four open-response items. Third grade is presented first, followed by fourth grade. The results of each open-response item are presented in the following parts: Part 1, Open-Response Item; Part 2, Responses; and Part 3, Strategies (see Appendix A for coding keys; see Appendix B for frequencies of students' response- and strategy-type).

Third-Grade Chocolate Bar Task: Determining the Fractional Amount of a Region

Open-Response Item

The third-grade Chocolate Bar task assessed students' understanding of part-whole concepts (see Appendix A). The question presented students with a chocolate bar broken into four equal pieces with one piece eaten. Students determined the fraction of the original chocolate bar that was left (3/4), explaining their solution by drawing a picture. Figures 1 and 2 show examples of students' responses and strategies to the task.

Responses

The most common response type on the Chocolate Bar task for both VMR and PMTR groups was "Correct Drawing with Incorrect Solution of ¼." Figure 1 shows examples of this response type (see e.g., 1.a.1; 1.b.2; 1.c). This response demonstrated students' understanding of partitioning and naming a part of a whole, but

students did not answer the question, “What fraction of the original chocolate bar is left?” Therefore, these are misinterpretations of the task and not necessarily fraction misconceptions by students.

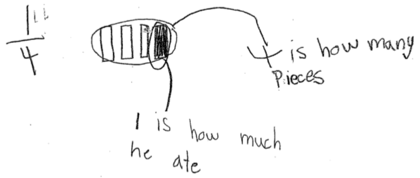
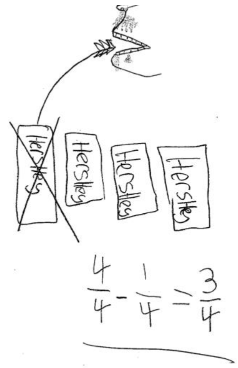
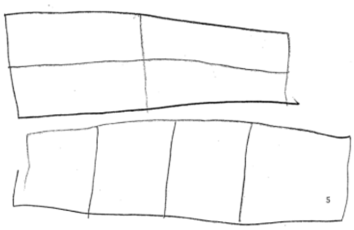
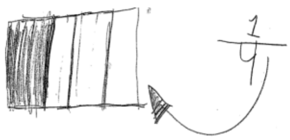
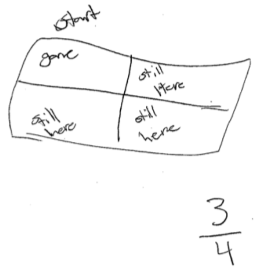
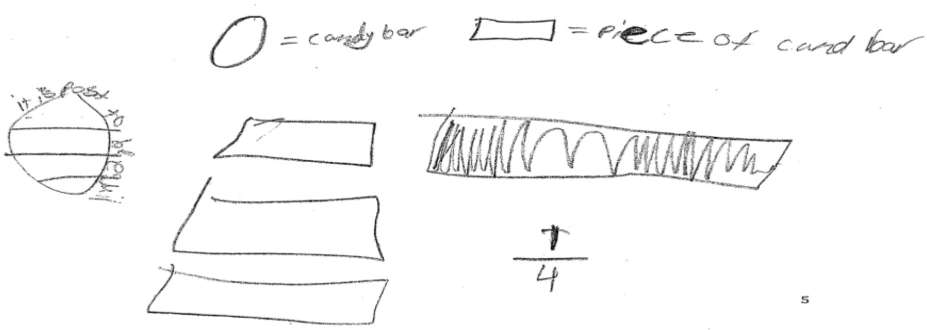
<table>
<tr><td colspan="3">Strategy Used: 1 Set</td></tr>
<tr><td>1.a.1
Response Type: Correct Drawing with Incorrect Solution of ¼ (PMTR student)

</td><td colspan="2">1.a.2
Response Type: Correct (PMTR student)

</td></tr>
<tr><td colspan="3">Strategy Used: 2 Region</td></tr>
<tr><td>1.b.1
Response Type: Incorrect Drawing (VMR student)
</td><td>1.b.2
Response Type: Correct Drawing with Incorrect Solution of ¼ (PMTR student)

</td><td>1.b.3
Response Type: Correct (VMR student)

</td></tr>
<tr><td colspan="3">Strategy Used: 3 Both</td></tr>
<tr><td colspan="3">1.c
Response Type: Correct Drawing with Incorrect Solution of ¼ (PMTR student)

</td></tr>
</table>

Figure 1. Examples of Chocolate Bar Task response type codes grouped by strategy

Specifically, the example labeled 1.c showed that the PMTR student understood important fraction concepts, but did not answer the question. The 1.c student’s sentence about the region model stated, “it is [supposed] to be equal,” indicating that she understood that her region model (on the left) should show equal partitions and equal

pieces. The student's set model demonstrated her understanding that the candy bar was broken into four equal pieces and Jake ate one of those pieces. Her symbol of ¼ shows that she was linking her pictorial representation to a symbolic fraction notation. Despite demonstrating her understanding of these key concepts, the student did not respond with ¾ as the amount of the original chocolate bar that is left. This was common for many students in both instructional groups in Grade 3.

A larger percentage of PMTR students (84% PMTR students; 75% VMR students) created a correct drawing in their responses. These correct drawings provide insight into students' fraction understanding, and in some cases, their misinterpretations of the task. The incorrect drawings, on the other hand, provide insight into aspects of students' fraction understanding as well as into their fraction misconceptions. Example 1.b.1 shows a VMR student's response that was coded as "Incorrect Drawing." This drawing indicates that the student understood the need to partition a region into four equal parts, but he did not shade any of the partitions to represent a fractional amount.

Strategies

There were two main types of models students used for solving this open-response item: a region model and a set model. The context of the chocolate bar in this task encouraged students to draw a region model (see 1.b.1, 1.b.2, 1.b.3). However, the wording of the task, "Jake broke a chocolate bar into four equal pieces," provides an action that may have led students to draw a set model (thinking about the chocolate bar in individual pieces; see 1.a.1, 1.a.2).

The most common strategy for solving the Chocolate Bar task in both the VMR and PMTR groups was the use of a region model; however, more VMR students used the region model (81% of VMR students; 67% of PMTR students). More PMTR students (22%) used the set model for solving the Chocolate Bar task than did VMR students (9%). Figure 1 illustrates examples of students' use of these two types of models. Example 1.c showed one student's use of both models, highlighting different aspects of the student's conceptual understanding of fractions. The region model shows her understanding of partitioning while the set model highlights her understanding of shading 1 out of 4 pieces and linking that representation to the notation ¼.

Third-Grade Candy Cane Task: Determining the Fractional Amount of a Set

Open-Response Item

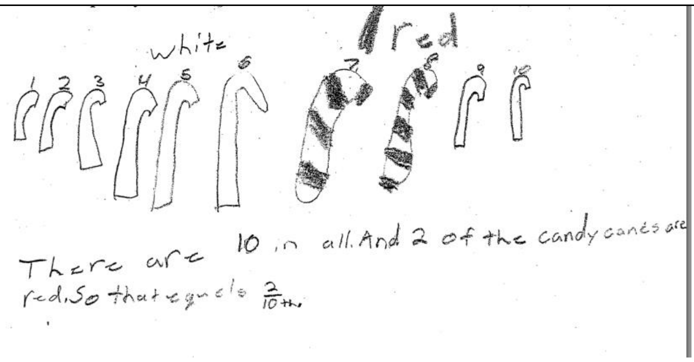
The Candy Cane task asked third-grade students to determine a fractional amount of a set of 10 candy canes. The pretest asked students to determine one-fifth of the set of 10 candy canes while the posttest asked students to identify two-fifths of the set. This task required that students understand that the relationship of red candy canes to the total number of candy canes was two out of five (posttest), to understand that two groups of five consisted of 10 candy canes, and to determine that if each group of five candy canes consisted of two red candy canes then two groups of five would have 4 candy canes.

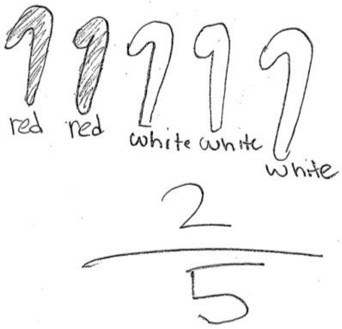
Responses

The majority of students' responses on the posttest contained three types of errors. Figure 2 provides an example of each type of error: one-half, drew 2/5, and drew 2/10. The most common error for both groups was "Drew 2/10." Example 2.a shows that the PMTR student understood he needed 10 candy canes ("There are 10 in all."), but as he worked to determine how many candy canes to color red, he focused only on the numerator (of 2/5) and did not recognize that there must be two *groups* of five candy canes ("And 2 of the candy canes are red.") Thus, he responded with 2 of the 10 candy canes being red ("So that equals $2/10^{th}$."). This type of error made up 37% of the PMTR group's responses and 27% of the VMR group's responses.

Another response-type error was coded as "Drew 2/5." Example 2.b shows that the VMR student focused on the phrase "2/5 of the candy canes were red" and drew only five candy canes with 2 being red. This error differed from the Drew 2/10 error. Rather than focusing on only the numerator as a whole number, the student drew a picture that represented a fraction. This type of error made up 10% of the PMTR group's responses and 21% of the VMR group's responses.

<table>
<tr><th colspan="2">Response Types</th></tr>
<tr><td>2.a

Response Type: Drew 2/10
Strategy: Start with 10
(PMTR student)</td><td>
</td></tr>
<tr><td>2.b

Response Type: Drew 2/5
Strategy: Start with 5
(VMR student)</td><td>
</td></tr>
<tr><td>2.c

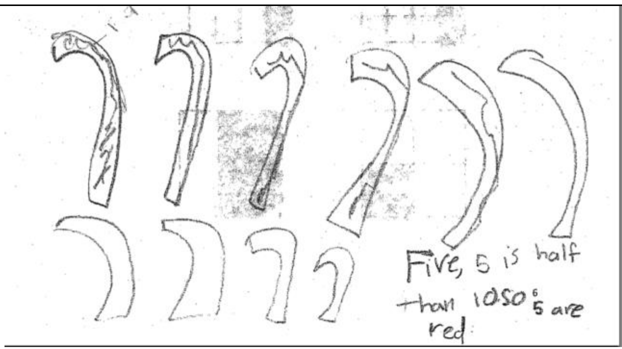
Response Type: One-Half
Strategy: Start with 10
(VMR student)</td><td>
</td></tr>
</table>

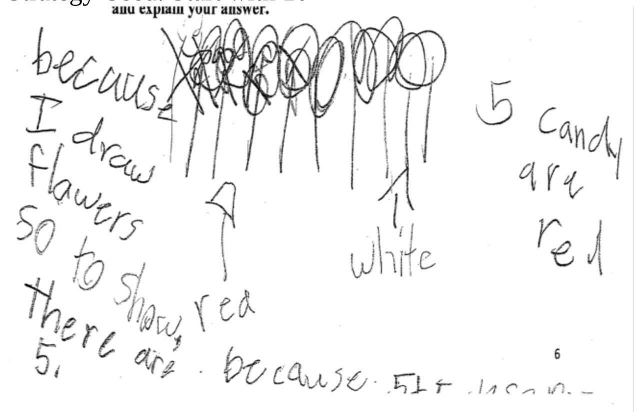
Figure 2. Examples of the three response types showing students' errors on the Candy Cane task

Example 2.c illustrates the "One-Half" response type. In this example, the student responded to the Candy Cane task with a drawing representing one-half. The student drew 10 candy canes and focused on the denominator of 5 within 2/5 to determine that she should color 5 of the canes red. While incorrect, this approach shows some relational thinking in that the student is considering a fractional amount of 10. This error made up 11% of the PMTR group's responses and 20% of the VMR group's responses.

Although students' correct responses remained low on the Candy Cane task posttest, there were many observable changes in students' responses and strategies from pretest to posttest. However, these responses often changed from one type of error to another type of error. Nevertheless, even the change in errors provides insight into students' fraction conceptions. Figure 3 shows an example of one student's pretest and posttest responses and strategies for the Candy Cane task.

The student in Figure 3 responded with the "one-half" response on the pretest, explaining that five are white and five are red because 5 + 5 equals 10. Notice that a symbolic fraction is not included in her response. On the posttest, she again drew 10 candy canes, but this time considered the fraction "two-fifths," as illustrated by her circling of five candy canes, denoting 2 of the 5 as red, and including the symbolic notation 2/5. Her question mark could be an indicator that she knows something is not correct, but is not sure. Overall, this student's posttest provides a window into changes in her understanding of fraction concepts. The posttest response shows that she can identify, draw, and symbolically represent a fraction (i.e., 2/5), whereas her pretest did not reveal

this understanding. Rather, the pretest response only showed that she possibly knew half of 10 is 5 and pinpointed a relationship between 10 and 5.

Pretest (VMR student) Response Type: One-half Strategy Used: Start with 10	Posttest (VMR student) Response Type: Drew 2/5 Strategy Used: Start with 10


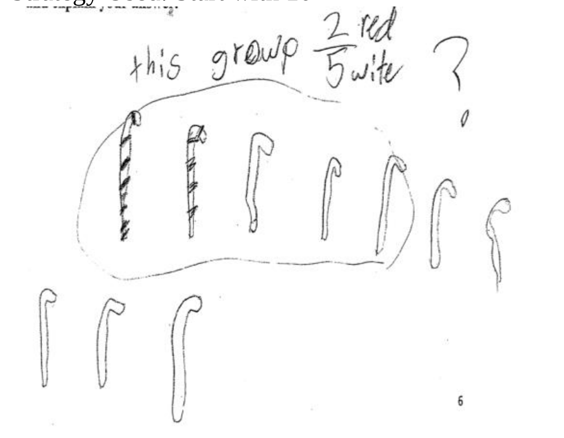


Figure 3. Example of a student's pretest and posttest responses on the Candy Cane task.

Strategies

The "Start with 10" strategy (2.a in Figure 2 and Figure 3) was the most commonly used strategy on the Candy Cane task (55% of the VMR group's strategies; 62% of the PMTR group's strategies). Typically, the "Start with 10" strategy was tied to the "Drew 2/10" response and the "One-Half" response, as seen in example 2.a in Figure 2 and Figure 3 respectively, which accounts for the predominant use of this strategy.

Fourth-Grade Comparing Fractions Task: Evaluating a Comparison of Two Fractions

Open-Response Item

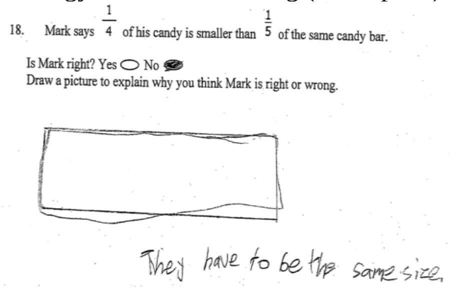
The Comparing Fractions task required fourth-grade students to evaluate a comparison of two fractions presented in a region model context (candy bar) then draw a picture to justify their evaluation of the comparison. The pretest focused on the comparison of unit fractions (1/4 and 1/5) while the posttest asked students to compare two fractions close to one whole (2/3 and ¾). On the pretest, students were asked to evaluate if Mark is correct in saying that ¼ of his candy bar is smaller than 1/5 of the same candy bar. On the posttest, students determined if Mark is correct in saying that 2/3 of his candy bar is smaller than ¾ of the same candy bar.

Responses

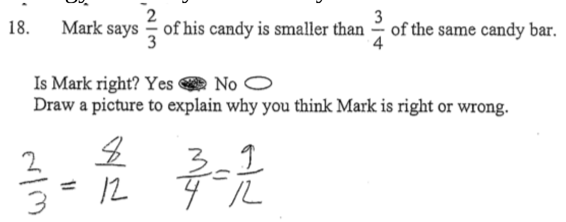
Overall, the most common response type for both groups was a correct response, and very few students' responses on the posttest fell within the error categories. The frequencies of the error patterns were nearly identical in both groups (see Appendix B). While these error types were not common on the posttest, it is interesting to see growth from pretest to posttest when incorrect solutions or error-types occurred on the pretest. Figure 4 provides two examples of students' growth between the pretest and posttest response.

The pretest for a VMR student (example 4.a.1) does not provide much information about what the student knows about fractions. There may even be a misconception related to treating fractions as whole numbers within the student's statement, "they have to be the same size." On the posttest (4.a.2) however, the student symbolically demonstrated some understanding of how to compare fractions. Example 4.b.1 shows a student's pretest response focused on comparing just the numerators ("He still just got one piece") rather than thinking about fractional amounts. While the posttest (4.b.2) shows this student's continued focus on whole numbers ("eight is smaller than nine"), she is now drawing models that are labeled with fractional amounts.

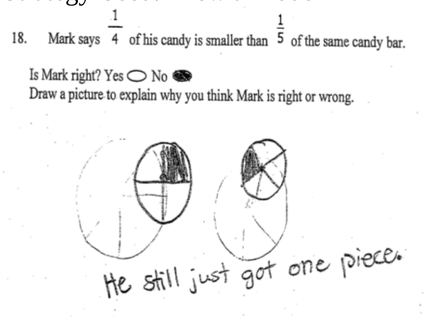
4.a.1
Pretest (VMR student)
Response Type: Incorrect
Strategy Used: No Drawing (incomplete)



4.a.2
Posttest (VMR student)
Response Type: Correct
Strategy Used: Symbolic Only



4.b.1
Pretest (PMTR student)
Response Type: Missing One Piece
Strategy Used: Drew a Model



4.b.2
Posttest (PMTR student)
Response Type: Correct
Strategy Used: Drew a Model

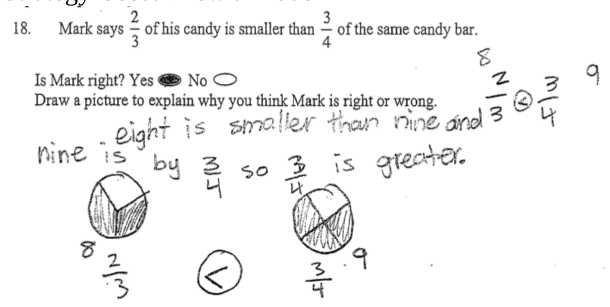


Figure 4. Examples of students' pretest and posttest response types and strategies for the Comparing Fractions task

Strategies

The two main strategies for comparing 2/3 and ¾ were to do so symbolically or through drawing models of the fractions. Overall, most students (75% of VMR; 82% of PMTR students) drew some type of representation to help them solve the task.

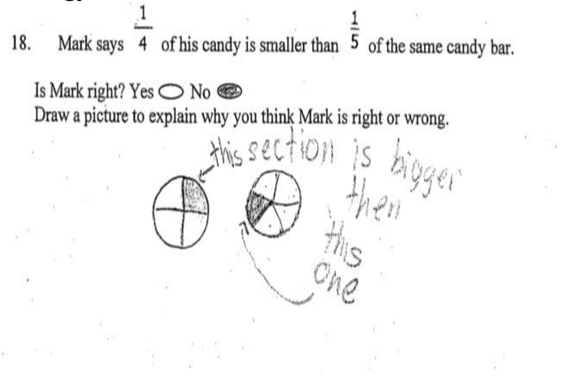
While "Drew a Model" was the most common strategy for this task, it is interesting to see the way that the use of this strategy changed from pretest to posttest for many students. Figure 5 illustrates three examples of students' pretests and posttests using the "Drew a Model" strategy.

If we only look at students' correct and incorrect answers, it appears that these three students did not exhibit change in their fraction learning from pretest to posttest. However, a qualitative analysis provides deeper insight into students' knowledge gains. Example 5.a.1 shows a student's correct response and use of a circle model for solving the task. Her sentences explain that the ¼ piece is bigger than the 1/5 piece. This same student also responded correctly on the posttest (see 5.a.2) and again drew a model, but this time she used a rectangular model. The way she lined up the two rectangular models indicated her clearer understanding of comparing two fraction representations. On the pretest she focused on comparing the partitioned pieces. On the posttest, her representations show her knowledge of comparing fractions with the same size whole. Her dotted line from the 2/3 model to the ¾ model allows us to infer that she understands she is comparing the shaded regions.

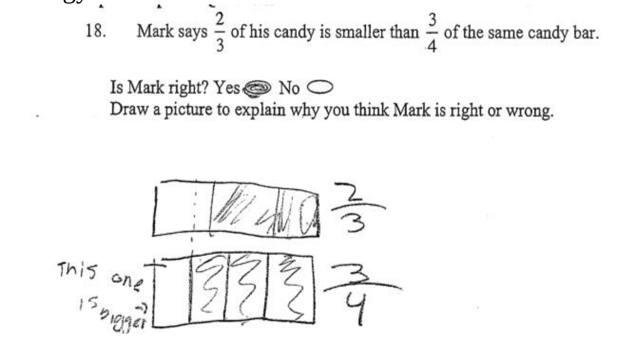
Examples 5.b.1 and 5.b.2 show a student's "Correct Drawing, Missing Explanation" on both the pretest and posttest. This student used the "Drew a Model" strategy to solve both tasks, but note the difficulty with partitioning rectangles in 5.b.1 (see erased rectangles) and ease of partitioning the circles on the posttest in 5.b.2. The other notable difference from pretest to posttest is the student's use of symbolic fraction notation representing each partitioned piece on the posttest. Similar to student 5.a's drawings, student 5.b's drawings and

notations show growth in demonstrating understanding, despite the test score remaining the same from pretest to posttest.

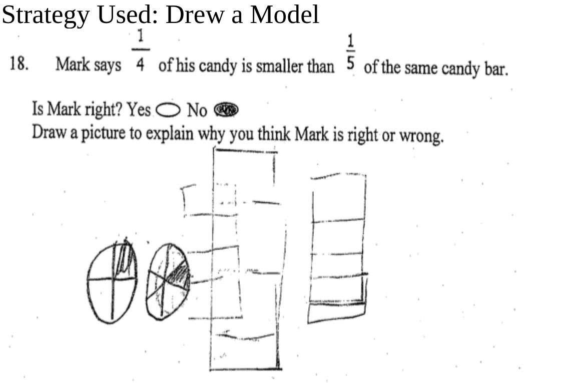
5.a.1
Pretest (PMTR student)
Response Type: Correct
Strategy Used: Drew a Model



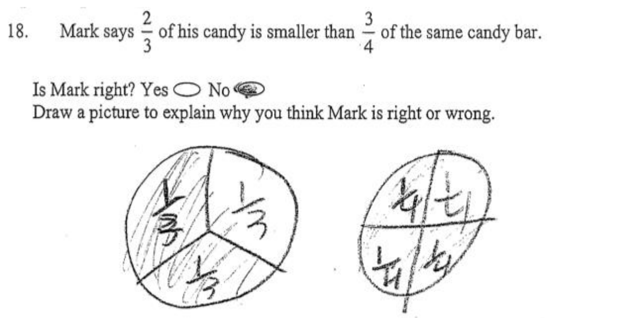
5.a.2
Posttest (PMTR student)
Response Type: Correct
Strategy Used: Drew a Model



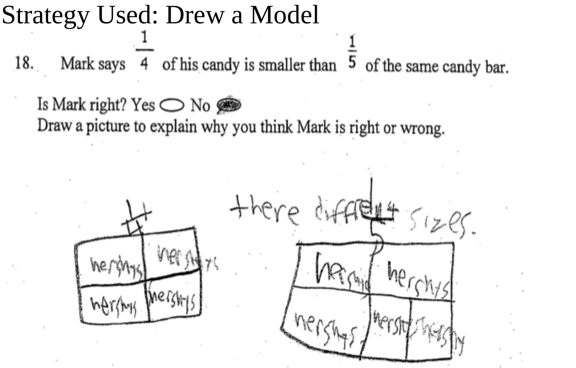
5.b.1
Pretest (VMR student)
Response Type: Correct Drawing, Missing Explanation
Strategy Used: Drew a Model



5.b.2
Posttest (VMR student)
Response Type: Correct Drawing, Missing Explanation
Strategy Used: Drew a Model



5.c.1
Pretest (PMTR student)
Response Type: Incorrect
Strategy Used: Drew a Model



5.c.2
Posttest (PMTR student)
Response Type: Incorrect
Strategy Used: Drew a Model

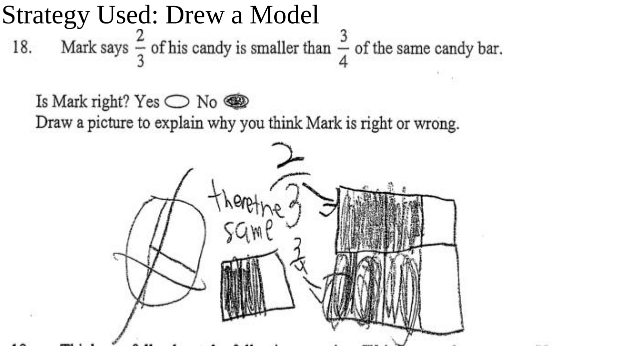


Figure 5. Examples of students' pretest and posttest response types and strategies for the Comparing Fractions task

Finally, student 5.c's "Drew a Model" code does not tell the whole story of his learning growth. On his pretest (5.c.1), the student drew different sized wholes and had difficulty partitioning the whole into equal pieces. While 5.c's test score remained constant on the posttest, the student's drawings and notations reveal newly

developed understandings. Example 5.c.2 shows two wholes of the same size with comparatively more accurate partitions of thirds and fourths. While there is marked growth in understanding in using the same size whole to compare fractions, the student continues to struggle with precise partitioning to help him solve the task accurately. Nevertheless, the manner of partitioning is improved.

Fourth-Grade String Task: Developing and Modeling Equivalent Fractions

Open-Response Item

The fourth-grade String task assessed students on representing fractions equivalent to one-half. The String task provided students with a series of representations equal to one-half (i.e., 3/6, ½, and 5/10) and asked students to partition a given whole into fractional amounts equal to one-half. The task provides insight into students' understanding of fractions equivalent to one-half, which is often their starting point for understanding equivalent fractions and comparing fractions. The wording for this task seemed to be confusing for students and likely impacted students' responses.

Responses

Overall, the percentage of correct responses was similar in both groups (45% of the VMR group; 43% of the PMTR group). Many students demonstrated correct responses on the pretest, too, however, their symbolic notations tied to their drawings and/or their explanations often provided a glimpse into solidified or new fraction conceptions. Examples 6.a.1 and 6.a.2 show a student's "Correct" response on both the pretest and posttest. On the pretest the student renames 5/10 and 11/22 as ½ and 3/6 to prove her solution. On the posttest, rather than just listing equivalent fractions, she used the equal sign and further explained that 11/22 is equal to the examples in the test question (5/10, 3/6, and ½).

While 7.a's test score on this question did not change from pretest to posttest, the inclusion of the equal sign and her written reasoning provide more insight into her fraction understanding and show more precision in her response. Similarly, examples 6.b.1 and 6.b.2 show a "Correct" response on both the pretest and posttest. The student's sentence on the posttest, "These are the same except there in smaller pieces," explains his understanding that equivalent fractions on this task are the same shaded region, no matter how many partitions are used to cut up that region.

Some students' responses showed correct thinking in the context of a misinterpretation of the task. Examples 6.c.1 and 6.c.2 provide an instance of a misinterpretation of the question on the pretest, followed by a correct interpretation on the posttest. Even within this misinterpretation of the question on the pretest, the student's explanation revealed some understanding of equivalent fractions.

The student's "Technically Correct" response on the pretest (see 6.c.1) overlooks her correct conception that amounts can be equivalent even if they do not visually look the same or are shaded in the same way. Her response on the posttest showed a correct interpretation of the question and revealed more information about her conception of equivalent fractions. This student's response on the posttest (see 6.c.2) revealed new or solidified understandings of equivalent fractions (i.e., all of these fractions are equal to ½) and also showed that she correctly understood the question.

Strategies

Examples 6.a.1 and 6.a.2 show the "Pieces on Each Side" strategy while examples 6.b.1 and 6.b.2 show the "Partitioned One Side" strategy. The "Pieces on Each Side" strategy was the most common strategy used on the String task (50% of VMR students; 56% of PMTR students) and this strategy most often led to a correct response. When students used some other strategy, such as a different model, this most often led them to an incorrect response.

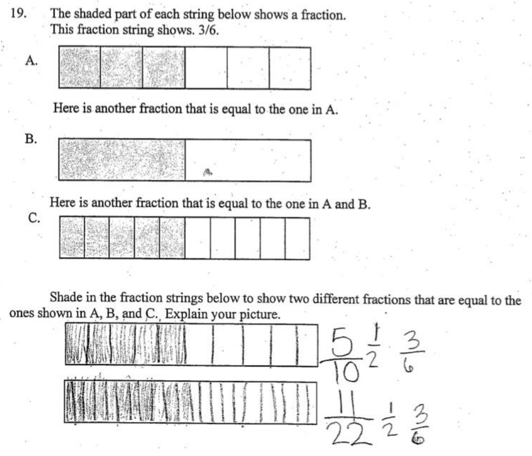
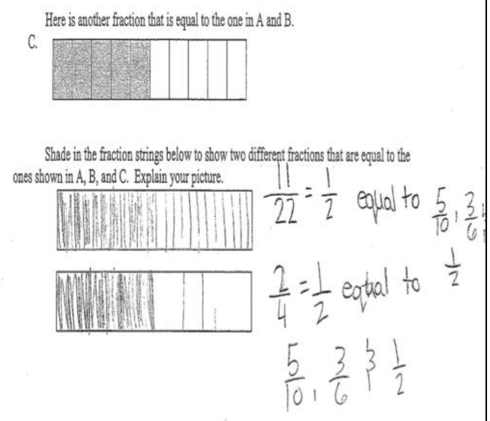
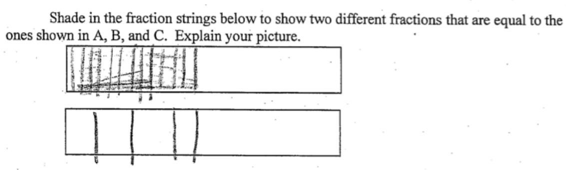
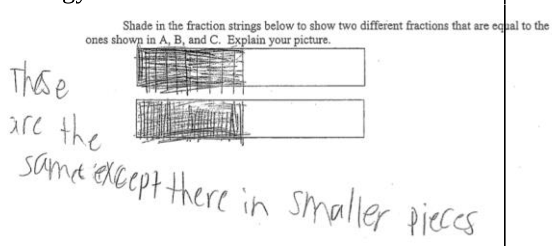
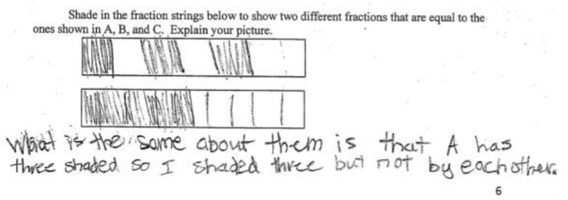
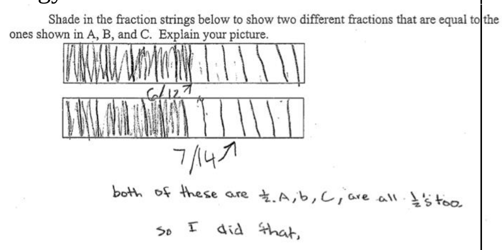
6.a.1 Pretest (PMTR student) Response Type: Correct Strategy Used: Pieces on Each Side  	6.a.2 Posttest (PMTR student) Response Type: Correct Strategy Used: Pieces on Each Side 
6.b.1 Pretest (VMR student) Response Type: Correct Strategy Used: Partitioned One Side  	6.b.2 Posttest (VMR student) Response Type: Correct Strategy Used: Partitioned One Side 
6.c.1 Pretest (VMR student) Response Type: Technically Correct Strategy Used: Pieces on Each Side  	6.c.2 Posttest (VMR student) Response Type: Correct Strategy Used: Pieces on Each Side 

Figure 6. Examples of students' pretest and posttest response types and strategies for the Strings task.

Discussion

The purpose of this study was to investigate relationships between instructional modalities for learning fraction concepts and students' responses and strategies for fraction tasks post-treatment instruction. VMR and PMTR instruction led students to use similar responses and strategies on the open-response fraction items. Overall, the results of this study corroborate previous research indicating the importance of both types of instructional modalities, showing that virtual manipulatives and physical manipulatives are effective instructional tools with positive effects on student learning (e.g., Burns & Hamm, 2011; Carbonneau et al., 2012; Mendiburo & Hasselbring, 2011; Moyer-Packenham & Westenskow, 2013; Sowell, 1989). However, a conversion mixed methods analysis offers researchers an opportunity to explore the nuances of students' fractions understanding. The results of this study provide a window into students' shifts in fraction learning after 2-3 weeks of instruction.

Third-Grade Open-Response Items

Drawing a Fractional Amount

The third-grade open-response fraction items assessed students' abilities to draw and name fractional amounts of a region and a set. Students' responses to the Chocolate Bar task showed that the PMTR group had a higher percentage of correct drawings (84% of PMTR students; 75% of VMR students). This small percentage difference may have resulted because the VMR groups did not have as much time as the PMTR groups to practice physically drawing the fraction models because they spent more time with models on the computer. The VMR group may have needed more opportunities to draw fraction models or translate their understandings to pictorial representations. It is also possible that the more time students spend drawing fraction models, the more they interact with the concept of the whole. In contrast, the VMR applets provide the whole; hence, it could be possible that the students in the VMR group did not need to consider the whole when using the applets. Maintaining conceptualization of the whole is critical to students' development of fraction understandings (Behr et al., 1983), and it is possible that VMR students did not have as many opportunities to develop this key idea since the whole was already provided in the applets.

There was also a difference in VMR and PMTR students' pictorial representations on the Chocolate Bar task. Twenty-two percent of the PMTR group's strategies were based on the set model, while only 9% of the VMR group used a set model to solve the task. This result could have been influenced by the predominant use of region models in the virtual manipulatives applets. Dienes' (1973) notion of embodied knowledge supports the interpretation of this study's results indicating that students learn abstract fraction concepts by interacting with multiple embodiments and making connections among representations of these concepts. If students lacked opportunities to draw representations, interact with the concept of the whole, or make connections between set and area models, this lack of opportunity to interact with the concept in one embodiment or another sometimes manifested in a student's posttest solution strategy.

Errors that Show Understanding

Both the VMR and PMTR groups performed poorly on the Candy Cane task, likely due to the difficulty level this task entailed for third-grade students. Nonetheless, students' errors on this task provide insight into their thinking and understanding of fractions. The errors on the Candy Cane task showed that PMTR students paid more attention to the whole (i.e., 10 candy canes) while the VMR students paid more attention to a fractional amount named in the task (i.e., 2/5). An analysis of the errors for the Candy Cane task showed that more PMTR students made the error of "Drew 2/10" (37% of the PMTR group's responses versus 27% of the VMR group's responses) while the VMR group more often made the "One-Half" error and the "Drew 2/5" error (20% VMR versus 11% PMTR; 21% VMR versus 10% PMTR). This difference in errors gives insight into students' conceptual understanding of fractions.

For example, students who drew 10 candy canes and then colored 2 candy canes, were likely thinking in terms of whole numbers, not fractional amounts. The task states "10 candy canes" and "two-fifths of the candy canes are red;" hence, students drew 10 candy canes and focused on the whole number "two" in "two-fifths" to represent 2 red candy canes. In contrast, students who drew 2/5 were likely thinking in terms of fractions because they drew a model that represented a fractional amount presented in the task—the fraction of 2/5. They understood the fractional relationship of 2/5 and could model it, but did not use that model to extend it to 10 candy canes. Instead those who drew 10 candy canes but represented 2 as red focused more on the 2 as a whole number rather than the relationship of the numbers expressed by the fraction. Similarly, the "One-Half" error signals some understanding of a fractional amount or part of a whole, as students saw the 10 candy canes as the whole and knew half that amount is 5 (using the 2 or the 5 in 2/5).

In sum, it is possible that students in the VMR group exhibited a stronger conceptual understanding of the meaning of fractional amounts as part-whole relationships than the PMTR group based on the errors by students in the Candy Cane task. The PMTR group may have had more understanding of the whole, in that their drawing needed to have ten candy canes. Both concepts are difficult for children to develop as they learn fractions (Behr et al., 1983; Smith, 2002), and these results suggest that VMR and PMTR modalities hold affordances for highlighting the "part-whole relationship" concept and the "maintaining the whole" concept in different ways. The instructional implication is that students need opportunities to interact with these concepts using varied modalities and embodiments of those concepts.

Fourth-Grade Open-Response Items

Shifts in Learning

The fourth-grade open-response fraction items assessed students on comparing two fractions (specifically, 2/3 and ¾) and naming and partitioning fractions equivalent to ½. The pretest and posttest examples in the Results section were chosen to highlight how test scores can conceal students' new or solidified fraction understandings post-instruction. A qualitative analysis of students' responses and strategies on these open-response items reveal important shifts in fraction learning, despite a student's score remaining constant. These shifts in learning occurred in both the VMR and PMTR groups. Both concepts—comparing fractions and equivalent fractions—are easily visible when assessed with open-response items. However, there are fraction sub-concepts and skills that students need before they can successfully understand comparing fractions and equivalent fractions, which are not readily visible through a quantitative analysis. For example, Westenskow and Moyer-Packenham (in press, 2016) developed an "iceberg model" for equivalent fractions in which the tip of the iceberg above the water line represents equivalent fractions understanding while sub-concepts such as naming, modeling, identifying, and comparing fractions sit below the water line and are necessary skills for equivalent fraction understanding.

The examples of students' pretest and posttest response types and strategies for the Comparing Fractions and String tasks highlight these shifts in learning that occur below the water line. "Drew a Model" was the most common strategy for the Comparing Fractions task and the pretest-posttest examples in Figure 5 illustrate the shifts in learning that occurred below the water line. Some students' responses indicated no numerical change from pretest to posttest, however drawings revealed clearer understanding of comparing two fraction representations, specifically, the important concept of comparing fractions with the same size whole. Similarly, the pretest-posttest example provided in Figure 6 for the String task showed a student's shift in learning as she used symbols and words to more precisely describe an equivalent relationship. Her score on this open-response item remained constant from pretest to posttest, but a qualitative look at her drawings highlighted a shift in understanding the concepts below the tip of the iceberg for equivalent fraction understanding.

Implications and Recommendations

Solution Representations

An analysis of students' responses and strategies through their solution representations (Cai, 2005) provided insight into students' errors, thought processes, internal representations, abilities to model tasks in order to justify a solution, and strategies. Representations revealed that students in the third-grade VMR group needed more opportunities to draw representations during instruction. Students' representations also suggest that virtual manipulatives provide students with multiple opportunities to develop part-whole understandings of fractions and further their understandings of fractions as relationships. We also propose that students in the PMTR group had more opportunities to develop understandings about the whole and that the size of the whole is important. We learned that this is particularly difficult for students in the context of a set model.

Pedagogical Representations

Use of Virtual and Physical Manipulatives

Both virtual manipulatives and physical manipulatives lead to improvements in learning, but similar to previous research (Clements & McMillan, 1996; McNeil & Jarvin, 2007; Moyer, 2001) low numbers of correct responses cause us to pause about *how* instructional modalities are being used. If virtual or physical manipulatives are only used to teach procedures, students may continue to struggle with understanding fractions, even with representations present in the instruction (Moss & Case, 1999). In her study on how teachers use physical manipulatives, Moyer (2001) explained that students must understand the representation in order for it to aid their understanding of the concept. This, too, applies to virtual manipulatives. Based on teacher surveys, many students likely had not interacted with virtual manipulative tools prior to the study and possibly needed more time with the computer, tools, or various exploratory experiences with virtual manipulatives to understand the representations presented in the tools.

Making Connections

A beneficial quality of many virtual manipulatives applets is their simultaneous linking of pictorial and symbolic representations (Moyer-Packenham & Westenskow, 2013). Physical manipulatives do not typically contain the simultaneous linking affordance; therefore, many researchers have described the importance of transparency and linking representations when using physical manipulatives to teach concepts (Cramer, Post, & delMas, 2002). The results of this study further support these claims. On the Candy Cane task, errors that show fractional knowledge (versus errors that show students' reliance on whole number knowledge) may be the result of the simultaneous linking affordance of the virtual manipulatives applets. However, it is possible that students in the VMR group also needed more time to physically draw and partition the representations, which could account for the PMTR's better performance on drawing representations in the Chocolate Bar task. Furthermore, Dienes' notion of embodied knowledge provides a lens for considering the simultaneous linking features in virtual manipulatives as well as instructional strategies for providing students opportunities to interact with concepts within multiple embodiments as ways to improve the use of virtual and physical manipulatives in fraction instruction.

Future Studies

Our research lays the groundwork for future studies about how affordances of virtual manipulatives and physical manipulatives are useful in teaching specific fraction concepts. This study may lead to future examinations of how students develop their understanding of fractions as representations of the relationship between two quantities, rather than incorrectly interpreting fractions as representing two whole numbers. Another research extension could be exploring the affordances of drawing pictorial representations or using physical manipulatives with different sized wholes (e.g., pattern blocks) to develop students' abilities to maintain conceptualization of the whole.

This study contributes to a growing body of literature showing that virtual manipulatives and physical manipulatives are equally beneficial. Our results lead us to believe that a combination of instructional modalities are necessary in order for students to develop deep and connected understandings of rational numbers (e.g., Jordan & Baker, 2011). This type of analysis with a large sample of students complements both small-scale interview studies as well as large-scale quantitative studies. Our analysis of four open-response items for a large sample of students, combined with within-class random assignment, provides broad data about the importance of both virtual and physical manipulatives instructional modalities on students' fraction learning. Testing outcomes were the same for students using virtual manipulatives or physical manipulatives. At the same time, the complexity of such learning is maintained by observing and coding students' responses and strategies beyond the "correct" and "incorrect" codes typical of large-scale studies. Through an analysis of students' responses and strategies, we found errors and misconceptions that are common in the literature (e.g., Behr et al., 1983; Smith, 2002), VMR group success in understanding fractions as relationships, and PMTR group success in maintaining conceptualization of the whole.

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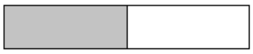
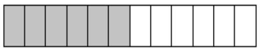
Appendix A

Overview of Open-Response Problems, Response Types Codes, and Strategies Used Codes

	Problem	Model	Big Ideas	Response Types	Strategies Used
Third Grade					
Chocolate Bar	*Jake broke a chocolate bar into four equal pieces and ate one piece. What fraction of the original chocolate bar is left? Explain using a picture that your answer is correct.*	Region	Partition a region, name a fraction (Part-whole understanding, identify fraction (1/4 and 3/4) of a whole)	1 Incorrect Drawing 2 Correct Drawing Only 3 Correct Drawing ¼ 4 Correct Drawing and Answer (3/4)	1 Set 2 Region 3 Both 0 Other
Candy Cane	*Sally has 10 candy canes. Two-fifth of the candy canes are red while the others are white. How many of the candy canes are red? Draw a picture and explain your answer.*	Set	Partition a set based on a given fraction and name number of objects (Fractional amount (2/5) of a set of 10 candy canes; Relationship btw whole number amount to total number of candy canes; Grouping)	1 No answer/Incomplete 2 One-half 3 Drew 2/5 4 Drew 2/10 5 Correct	1 Start with 5 2 Groups of 5 3 Start with 10 4 Other
Fourth Grade					
Comparing Fractions	*Mark says 2/3 of his candy is smaller than 3/4 of the same candy bar.* *Is Mark right? Yes No* *Draw a picture to explain why you think Mark is right or wrong.*	Region	Evaluate a comparison of two fractions (2/3 and ¾). Close to a whole Draw given fractions	1 Incorrect 2 Missing one piece 3 Subtracted two fractions 4 Compared numerator or denominator as whole numbers 5 Correct Drawing (explanation incomplete or missing) 6 Correct	1 Symbolic only 2 Drew a model 3 No drawing or no attempt
String	*The shaded part of each string below shows a fraction.*	Length	Partition a given whole	1 Incorrect	1 Partitioned One Side

This fraction string shows 3/6: A.  *Here is another fraction that is equal to the one in A* B.  *Here is another fraction that is equal to one in A and B.* C.  *Shade in the fraction strings below to show two different fractions that are equal to the ones shown in A, B, C. Explain your picture.*	Equivalent names for ½ (Developing and modeling equivalent fractions)	2 Technically Correct 3 Correct	2 Pieces on Each Side 3 Other 4 No Strategy

Appendix B

Percentage of Response Types and Strategies Used on the Third-Grade Chocolate Bar Problem

	VMR % (n = 73)	PMTR % (n = 82)
Response Types		
1 Incorrect Drawing	25%	16%
2 Correct Drawing Only	8	17
3 Correct Drawing with Incorrect Solution of ¼	42	37
4 Correct Drawing and Correct Solution of 3/4	25	30
Strategies Used		
1 Set	9	22
2 Region	81	67
3 Both	3	4
4 Other	7	7

Percentage of Response Types and Strategies Used on the Third-Grade Candy Cane Problem

	VMR % (n = 73)	PMTR % (n = 82)
Response Types		

1 No Answer/Incomplete	19%	24%
2 One-half	20	11
3 Drew 2/5	21	10
4 Drew 2/10	25	37
5 Correct	15	18
Strategies Used		
1 Start with 5	18	8
2 Groups of 5	11	15
3 Start with 10	55	62
4 Other	16	15

Percentage of Response Types and Strategies Used on the Fourth-Grade Comparing Fractions Problem

	VMR % (n = 94)	PMTR % (n = 106)
Response Types		
Incorrect	29%	26%
Missing One Piece	3	5
Subtracted Two Fractions	2	2
Compared As Whole Numbers	8	6
Correct Drawing, Missing Explanation	18	22
Correct	40	39
Strategies Used		
Symbolic Only	4	3
Drew a Model	75	82
No Drawing/Attempt	21	15

Percentage of Response Types and Strategies Used on the Fourth-Grade String Problem

	VMR % (n = 94)	PMTR % (n = 106)
Response Types		
Incorrect	34%	31%
Technically Correct	21	26
Correct	45	43
Strategies Used		
Partitioned One Side	7	5
Pieces on Each Side	50	56
Other	27	26
No Strategy	16	13